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THE SOLUTION OF CHAIN OF QUANTUM KINETIC EQUATIONS OF BOGOLIUBOV FOR BOSE SYSTEMS, INTERACTING BY DELTA POTENTIAL

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Abstract

The BBGKY's chain of quantum kinetic equations that describes the system of Bose particles interacting by delta potential is solved by the operator method with the help of nonlinear Schrödinger's equations. The solution of the chain is defined in terms of the Bethe ansatz.

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In this paper we consider a quantum system of Bose particles with the potential in the form of delta function [1]-[5]. Our purpose is to find the solution of the BBGKY [1] interconnected kinetic equations in terms of the Bethe ansatz.

We start from the nonlinear Schrödinger equation for the quantum field $\phi(t,x)$ [3]

$$i\frac{\partial}{\partial t}\phi(t,x) = -\frac{\partial^2}{\partial t^2}\phi(t,x) + 2\kappa\phi(t,x)^2\phi(t,x),$$

where x is the particle coordinate, $t \geq 0$ is the time, $\kappa > 0$.

In order to find a solution of the nonlinear Schrödinger equation by means of the standard approach, we consider an additional linear problem for this equation [6].

$$(i\frac{\partial}{\partial x} + \frac{1}{2}\xi)\phi_1 = -\sqrt{\kappa}\psi_2\phi, \tag{1}$$

$$(i\frac{\partial}{\partial x} - \frac{1}{2}\xi)\phi_2 = \sqrt{\kappa}\psi^*\phi_1, \tag{2}$$

with the boundary conditions for the Iost function $\psi(x) \to \infty$ for $x \to 0$ and

$$\psi(x,\xi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} exp(i\xi\frac{x}{2}) \qquad as \qquad x \to \infty$$

$$\chi(x,\xi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} exp(-i\xi\frac{x}{2}) \qquad as \qquad x \to \infty. \tag{3}$$

We introduce the transition coefficients

$$\psi(x,\xi) = \begin{pmatrix} \alpha(\xi)e^{i\xi\frac{x}{2}} \\ \beta(\xi)e^{-i\xi\frac{x}{2}} \end{pmatrix} \qquad for \qquad x \to \infty \tag{4}$$

which also hold in the quantum case [5], [3].

Equations (1),(2) and conditions (3)-(4) are equivalent to the integral equations

$$\psi_1(x,\xi) = e^{i\xi\frac{x}{2}}[1 + \sum_{s=1}^{\infty}\kappa^s\int_x^{\infty}...\int_x^{\infty}dx_1...dx_s dy_1...dy_s\Theta(x_1 < y_1 < ...x_s < y_s)$$

$$exp(i\xi(x_1 + ... + x_s - y_1 - ... - y_s))\phi^*(x_1)...\phi^*(x_s)\phi(y_1)...\phi(y_s)],$$

$$\psi_2(x,\xi) = -i\kappa e^{i\xi\frac{x}{2}}[\int dx_1 e^{i\xi x_1}\phi^*(x_1) + \sum_{s=1}^{\infty}\kappa^s\int_x^{\infty}...\int_x^{\infty}dx_1...dx_{s+1}dy_1...dy_s\theta(x_1 < y_1$$

$$< ...x_{s+1})exp(i\xi(x_1 + ...x_{s+1} - ... - y_1 - ... - y_s))\phi^*(x_1)...\phi^*(x_{s+1})\phi(y_1)...\phi(y_s)], \tag{5}$$

where $\theta(x) = 0$ for $x < 0, \theta(x) = 1$ for $x \geq 0$. Here $\phi(\xi), \phi^*(\xi)$ are the operators which satisfy the commutation relations

$$[\phi(\xi), \phi^*(\xi')] = \delta(\xi - \xi').$$

Within the context of equations (5) and condition (4) we have [3]

$$\alpha(\xi) = 1 + \kappa \int dx_1 dy_1 \theta(x_1 < y_1) exp(i\xi(x_1 - y_1))\phi^*(x_1)\phi(y_1) + +\kappa^2 \int ... \int dx_1 dx_2 dy_1 dy_2$$
$$\theta(x_1 < y_1 < x_2 < y_2) exp(i\xi(x_1 + x_2 + y_1 + y_2))\phi^*(x_1)\phi^*(x_2)\phi(y_1)\phi(y_2)....,$$
$$\beta(\xi) = \int dx_1 exp(i\xi x_1)\phi^*(x_1) + \kappa \int dx_1 dx_2 dy_1 \theta(x_1 < y_1 < x_2) exp(i\xi(x_1 + x_2 - y_1))$$
$$\phi^*(x_1)\phi^*(x_2)\phi(y_1) + ... \tag{6}$$

The eigenfunctions of the Hamiltonian $H = \int dx(\partial_1\phi^*\partial_1\phi + \kappa\phi^*\phi^*\phi\phi)$, may be obtained [3] by applying the operator β to vacuum $|0>$ $(\phi(x)|0>= 0)$

$$|k_1...k_s >= |\beta(k_1).....\beta(k_s)|0> . \tag{7}$$

Substituting (6) for $\beta(k)$ in (7) and performing some calculations, one finds this state to be equivalent to the well-known Bethe wave function

$$\int \prod_{i=1}^{\infty}[dx_i exp(ik_i x_i)][\prod_{1\leq i<j\leq s}(1 - \frac{i\kappa}{k_i - k_j}\varepsilon(x_i - x_j))]|\phi^*(x_1)...\phi^*(x_s)|0> . \tag{8}$$

However, the state (7), determined by the operator $\beta(k)$, is not normalised. Thus we introduce another operator $R^*(k)$ defined by

$$R^*(k) = \frac{\beta(k)}{\alpha(k)}.$$

Then the "*in*" and "*out*" states are defined for $k_1 < k_2 < ... < k_s$
as

$$|\phi(k_1...k_s) >_{in}= |R^*(k_1)...R^*(k_s)|0> \tag{9}$$

$$|\phi(k_1..k_s) >_{out}= |R^*(k_s)...R^*(k_1)|0> . \tag{10}$$

These states are normalised.
The operator $R^*(k)$ satisfies the commutation relation

$$R^*(k')R^*(k) = exp(2i\theta(k - k'))R^*(k)R^*(k'),$$

where

$$exp(2i\theta(k - k') = \frac{k - k' - i\kappa}{k - k' + i\kappa}.$$

Moreover,

$$R(k_i)R^*(k'_j) = exp(2i\theta(k_i - k'_j))R^*(k'_j)R(k_i) + 2\pi\delta(k_i - k'_j). \tag{11}$$

According to Ref.[3], the states (12),(13) can be proved to be identical to the states determined by the Bethe ansatz.

To do this, we introduce the Fourier transformation

$$R(x) = \int \frac{d\xi}{2\pi} exp(i\xi x) R(\xi)$$

and define the state $x_1....x_s >$ by

$$|x_1.....x_s >= |R^*(x_1)...R^*(x_s)|0 > .$$

Then, making use of the Gelfand-Levitan quantum equation [4]

$$\phi(x) = \int \frac{d\xi}{2\pi} R(\xi_1) exp(i\xi_1 x) + \kappa \int \frac{d\xi_1}{2\pi}\frac{d\xi_2}{2\pi}\frac{\xi_3}{2\pi}\frac{R^*(\xi_2)R(\xi_2)R(\xi_3)exp(i(\xi_1-\xi_2+\xi_3)x)}{(\xi_2-\xi_1-i\varepsilon)(\xi_3-xi_2+i\varepsilon)} + \ldots$$

we find that for $x > x_1$ and $i = 1, 2, ..., s$ we have

$$|\phi^*(x_1)...\phi^*(x_s)|0 >= |R^*(x_1)...R^*(x_s)|0 > .$$

It is not difficult to show that the state $|x_1, ..., x_s >$ indeed is determined by the Bethe ansatz.

We consider the wave function in the coordinate space

$$f(x_1, ..., x_s; k_1, ..., k_s) =< 0|\phi(x_1)...\phi(x_s)|k_1...k_s > .$$

As for Bose systems the wave function is symmetric in $x_1, ..., x_s$, sufficiently to consider it in the field $x_1 < x_2 < ..., x_3$. In this field

$$f(x_1, ..., x_s; k_1, ..., k_s) =< 0|R(x_1)...R(x_s)|k_1...k_s >= \int \prod_i (\frac{dp_i}{\pi} exp(ip_i x_i))\times$$
$$< 0|R(p_1)...R(p_s)|R^*(k_1)...R^*(k_s)|0 >,$$

which with the help of relation (11), gives the wave function in the Bethe ansatz.

Using the time dependence of operator

$$R^*(t, k) = (exp(-ik^2 t))R^*(0, k)$$

we can define the state $|t, x_1, ..., x_s >$ through the Bethe ansatz

$$|t, x_1, ..., x_s >= \int ... \int dk_1...dk_s f(t, x_1, ..., x_s; k_1, ..., k_s)\times$$

$$\varphi*(k_1,...,k_s)=\int\prod_i(\frac{dp_i}{2\pi}exp(ip_ix_i))exp(ik_i^2t)\times$$

$$<0|R(p_1)...R(p_s)|R^*(k_1)...R^*(k_s)|0>\varphi*(k_1,...,k_s)dk_1...dk_s. \quad (12)$$

Using formula (12) we can define the density matrix for particles:

$$F_s(t,x_1,...,x_s;x'_1,...,x'_s)=\int...\int\varphi(k'_1,...,k'_s)f^*(t,k'_1,...,k'_sx'_1,...,x'_s)\times$$

$$f(t,x_1,...,x_s;k_1,...,k_s)\varphi*(k_1,...,k_s)dk_1...dk_sdk'_1...dk'_s=\prod_i\int dk_idk'_i(\frac{dp_i}{2\pi}\frac{dp'_i}{2\pi}\times$$

$$\varphi(k'_1,...,k'_s)<0|R(k'_s)...R(k'_1)|R^*(p'_s)...R^*(p'_1)|0><0|R(p_1)...R(p_s)|\times$$

$$|R^*(k_1)...R^*(k_s)|0>\varphi*(k_1,...,k_s)exp(ip_ix_i-p'_ix'_i))exp(-i((k'_i)^2-k_i^2)t) \quad (13)$$

which is expressed by the Bethe ansatz.

The density matrix (13) is the solution of the chain of the Bogoliubov quantum kinetic equations describing the Bose systems, interacting by potential in the form of delta function in the domains of phase space where $x_k\neq x_l$ and $x'_k\neq x'_l$ for any $k\neq l$ [7],[8]:

$$i\frac{d}{dt}F_s(t,x_1,...,x_s;x'_1,...,x'_s)=(H(x_1,...,x_s)F_s(t,x_1,...,x_s;x'_1,...,x'_s)-$$

$$-F_s(t,x_1,...,x_s;x'_1,...,x'_s)H(x'_1,...,x'_s))+2\kappa\int\sum_{1\leq i\leq s}(\delta(x_i-x_{s+1})-\delta(x'_i-x_{s+1}))\times$$

$$\times F_{s+1}(t,x_1,...,x_s,x_{s+1};x'_1,...,x'_s,x_{s+1})dx_{s+1},$$

$$F_s(t,x_1,...,x_s;x'_1,...,x'_s)=\int F_{s+1}(t,x_1,...,x_s,x_{s+1};x'_1,...x'_s,x_{s+1})dx_{s+1};$$

where $F_s(t,x_1,...,x_s;x'_1,...,x'_s)$ is the density matrix, x is the particle coordinate, [,] is the Poisson bracket, 2m=1, $h=1$, $t\leq 0$ is the time, $\kappa\geq 0$, the Hamiltonian $H_s(x_1,...,x_s)$ is given by

$$H_s(x_1,...,x_s)=-\sum_{i=1}^{s}\frac{d^2}{dx_i^2}+2\kappa\sum_{i,j}\delta(x_i-x_j).$$

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